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Boundary Terms for Globally Supersymmetric Actions

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Supersymmetry is always broken on a manifold-with-boundary, since the supersymmetric variation of the Lagrangian is a total divergence which yields a boundary term when integrated. If attention is restricted to a subalgebra of generators which preserve the boundary, the invariance of the action can be restored by adding a boundary correction whose variation cancels that from the integrated Lagrangian. One can also impose boundary conditions which are invariant under this subalgebra.

1. INTRODUCTION

Supersymmetry occurs in a wide range of contexts, from field theory and string theory to stochastic processes. In many of these contexts, one wishes to calculate amplitudes between specified initial and final configurations. In the path-integral approach, the first step in this procedure is to define an action functional which depends on the configuration of the fields between these boundaries.

The action is usually expressed as the integral of a Lagrangian density, whose variation under supersymmetry transformations is a total divergence. When integrated, however, this variation gives rise to a boundary term which breaks the supersymmetry of the functional integral.

In some applications, the breaking of supersymmetry by boundaries is unimportant. Nonetheless, supersymmetric models often have useful or interesting properties, and it is natural to wonder whether the action can be modified in such a way that at least some of the supersymmetry is preserved.

If a symmetry is to be unbroken by the presence of boundaries, then all of its bosonic generators must themselves be symmetries of the boundary;

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hence, its algebra must not generate translations normal to the boundary. However, the full supersymmetry algebra contains all possible translations, including those normal to the boundary. One must therefore be content in this case to seek a subalgebra which excludes the generators of normal translations. The properties of this subalgebra are discussed in Section 2.

Assuming the existence of a boundary-preserving subalgebra, it is natural to look for some way to restore the invariance of the action under the reduced supersymmetry generated by this subalgebra. As shown in Section 2, this can be done by adding a total divergence to the super-Lagrangian.

One can also impose boundary conditions which transform in a well-defined manner under the boundary-preserving subalgebra, or which are invariant under this subalgebra. These possibilities are discussed in Section 3.

In this paper I consider only global (i.e., rigid) supersymmetry. The same methods can undoubtedly be generalized to the case of local supersymmetry, but the full analysis appears rather more complicated and has not yet been attempted. Although the following discussion refers only to simple supersymmetry, it generalizes trivially to extended supersymmetries with central charges.

2. AN INVARIANT ACTION

Suppose that the space $\mathcal{M}$ has coordinates x^μ ($\mu = 1, \ldots, n$) and a boundary $\partial\mathcal{M}$ with an outward-pointing unit normal 1-form $N_\mu\, dx^\mu$. We make no assumptions here about whether this 1-form is timelike or spacelike, although in many applications the boundary will represent an initial or final hypersurface.

Suppose that our model admits a global supersymmetry with fermionic generators Q_a ($a = 1, \ldots, D$) obeying the algebra

$$\{Q_a, Q_b\} = 2(C\gamma^\mu)_{ab}\partial_\mu, \qquad [Q_a, \partial_\mu] = 0 \tag{1}$$

where γ^μ are suitable gamma matrices and C represents the charge conjugation matrix. In superspace parametrized by commuting coordinates x^μ and anticommuting coordinates θ^a ($a = 1, \ldots, D$) these generators can be represented as

$$Q_a = \frac{\partial}{\partial\theta^a} + (C\gamma^\mu)_{ab}\theta^b\partial_\mu \tag{2}$$

Let us assume that there is a subalgebra of (1) which has $\hat{D}$ independent fermionic coordinates ($\hat{D} < D$) and $\hat{n}$ independent bosonic generators ($\hat{n} < n$); without loss of generality, we can take these to be $\{Q_{\hat{a}}: \hat{a} = 1, \ldots, \hat{D}\}$ and $\{\partial_{\hat{\mu}}: \hat{\mu} = 1, \ldots, \hat{n}\}$, respectively. In order for the subalgebra to preserve the boundary $\partial\mathcal{M}$, it is necessary that all of its bosonic generators $\partial_{\hat{\mu}}$ are symmetries of $\partial\mathcal{M}$. Consequently, they must all be tangential to $\partial\mathcal{M}$ and perpendicular to the normal 1-form $N_\mu\, dx^\mu$; hence $N_{\hat{\mu}} = 0$.

In our notation, Latin indices with carets will always run from 1 to $\hat{D}$, and Greek indices with carets from 1 to $\hat{n}$. Similarly, barred Latin indices will run from $\hat{D} + 1$ to D, and barred Greek indices will run from $\hat{n} + 1$ to n.

To ensure that the subalgebra closes, we require that any anticommutator $\{Q_{\hat{a}}, Q_{\hat{b}}\}$ of fermionic generators is spanned by the bosonic generators $\partial_{\hat{\mu}}$. There must be no contribution from the bosonic generators $\partial_{\bar{\mu}}$ outside the subalgebra,

$$(C\gamma^{\bar{\mu}})_{\hat{a}\hat{b}} = 0 \tag{3}$$

and so one obtains the identities

$$\{Q_{\hat{a}}, Q_{\hat{b}}\} = 2(C\gamma^{\hat{\mu}})_{\hat{a}\hat{b}}\partial_{\hat{\mu}}, \qquad [Q_{\hat{a}}, \partial_{\hat{\mu}}] = 0 \tag{4}$$

We now seek an action which is invariant under the subalgebra (4). The problem is that the generators $Q_{\hat{a}}$ still contain derivatives normal to the boundary; consequently the supersymmetric variation of any super-Lagrangian will be a total divergence, whose integral will produce nonvanishing boundary terms which spoil the invariance.

This is seen most clearly in terms of the decomposition

$$Q_{\hat{a}} = \hat{Q}_{\hat{a}} + K_{\hat{a}} \tag{5}$$

where

$$\hat{Q}_{\hat{a}} = \frac{\partial}{\partial\theta^{\hat{a}}} + (C\gamma^{\hat{\mu}})_{\hat{a}\hat{b}}\theta^{\hat{b}}\partial_{\hat{\mu}} \tag{6}$$

and

$$K_{\hat{a}} = (C\gamma^{\mu})_{\hat{a}\bar{b}}\theta^{\bar{b}}\partial_{\mu} \tag{7}$$

The action of Q_a on a super-Lagrangian $\mathcal{L}$ now consists of two parts, $\hat{Q}_{\hat{a}}\mathcal{L}$ and $K_{\hat{a}}\mathcal{L}$. The first of these is made up of derivatives with respect to the coordinates $x^{\hat{\mu}}$ and $\theta^{\hat{a}}$, which are unbounded in the sense that $\partial_{\hat{\mu}}$ and $\partial_{\hat{a}}$ are tangential to the boundary. Consequently, the integral $\hat{Q}_{\hat{a}}\mathcal{L}$ over these coordinates vanishes. However, integrating $K_{\hat{a}}\mathcal{L}$ over all the coordinates yields a boundary term, since some of the derivatives ∂_μ have components normal to the boundary.

The generators $\hat{Q}_{\hat{a}}$ commute with the coordinates $x^{\bar{\mu}}$ and anticommute with the coordinates $\theta^{\bar{a}}$. Moreover, they obey the familiar relations

$$\{\hat{Q}_{\hat{a}}, \hat{Q}_{\hat{b}}\} = 2(C\gamma^{\hat{\mu}})_{\hat{a}\hat{b}}\partial_{\hat{\mu}}, \qquad [\hat{Q}_{\hat{a}}, \partial_{\hat{\mu}}] = 0 \tag{8}$$

For these reasons, it is natural to think of the $\hat{Q}_{\hat{a}}$ as generators of a "reduced" supersymmetry which acts in a superspace with coordinates $(c^{\hat{\mu}}, \theta^{\hat{a}})$. (Of course, whether this is a genuine supersymmetry depends on whether the algebra can be enlarged to include the generators of rotations. However, this question does not concern us here.) The remaining coordinates $(x^{\bar{\mu}}, \theta^{\bar{a}})$ are just passive spectators of this reduced supersymmetry.

Because the generators $\hat{Q}_{\hat{a}}$ obey the same anticommutation relations as the $Q_{\hat{a}}$, the reduced supersymmetry can be thought of as an alternative representation of the boundary-preserving subalgebra. Since unwanted boundary terms arise from the variation of any super-Lagrangian transforming in the original representation (4), we might seek instead a modified super-Lagrangian $\mathcal{L}'$ which transforms in the new representation (8). The integral of $\hat{Q}_{\hat{a}}\mathcal{L}'$ over the coordinates $x^{\hat{\mu}}$ and $\theta^{\hat{a}}$ would then be guaranteed to vanish, unlike the integral of $Q_{\hat{a}}\mathcal{L}$. If we also demand that $\mathcal{L}'$ and $\mathcal{L}$ should differ by a total divergence, then the dynamics of the model will be unaffected.

Under an infinitesimal boundary-preserving transformation $\mathcal{L} \mapsto \mathcal{L}_\epsilon = (1 + \epsilon^{\hat{a}} Q_{\hat{a}})\mathcal{L}$ the modified super-Lagrangian must transform as

$$\mathcal{L}' \mapsto \mathcal{L}'_\epsilon = (1 + \epsilon^{\hat{a}} \hat{Q}_{\hat{a}})\mathcal{L}' \tag{9}$$

It is not difficult to show that the only satisfactory choice is

$$\mathcal{L}' \equiv \exp(\theta^{\hat{a}} K_{\hat{a}})\, \mathcal{L} \tag{10}$$

The exponential is defined by a power series which terminates after a finite number of terms, owing to the properties of Grassmann variables. Using the anticommutation rules for $\hat{Q}_{\hat{a}}$, $K_{\hat{a}}$, and $\theta^{\hat{b}}$, it is a simple exercise to show that

$$\exp(\theta^{\hat{a}} K_{\hat{a}})\, \epsilon^{\hat{a}} Q_{\hat{a}} = \epsilon^{\hat{a}} \hat{Q}_{\hat{a}} \exp(\theta^{\hat{a}} K_{\hat{a}}) \tag{11}$$

The desired result (9) then follows immediately.

In fact $\theta^{\hat{a}} K_{\hat{a}}$ is a linear combination of the derivatives ∂_μ (with nilpotent commuting Grassmann coefficients) and so the necessary modification of $\mathcal{L}$ can be thought of as a kind of translation.

If we write the components of $\mathcal{L}$ as

$$L = \int d^D\theta\, \mathcal{L}, \qquad L^a = \int d^D\theta\, \theta^a \mathcal{L}, \qquad L^{ab} = \int d^D\theta\, \theta^a \theta^b \mathcal{L}, \ldots \tag{12}$$

then the correction to the L component, which represents the ordinary Lagrangian of the theory, is found to be

$$\delta L \equiv \int d^D\theta \, (\mathscr{L}' - \mathscr{L})$$
$$= \frac{1}{1!} (C\gamma^\mu)_{\hat{a}\bar{b}} \partial_\mu L^{\hat{a}\bar{b}} + \frac{1}{2!} (C\gamma^\mu)_{\hat{a}\bar{b}} (C\gamma^\nu)_{\hat{c}\bar{d}} \partial_\mu \partial_\nu L^{\hat{a}\bar{b}\hat{c}\bar{d}} + \cdots \quad (13)$$

The total action can now be written

$$S \equiv \int_{\mathcal{M}} d^n x \, d^D\theta \, \mathscr{L}' = \int_{\mathcal{M}} d^n x \, (L + \delta L) \quad (14)$$

By construction, this expression is completely invariant under the boundary-preserving subalgebra [as represented by (4) on ordinary superfields such as $\mathscr{L}$, or by (8) on $\mathscr{L}'$]. Moreover, since δL is a total divergence, it can be integrated to give a boundary correction. Consequently, its presence does not affect the dynamics of the model.

At this stage it is worth noting that (13) is the *only* admissible correction to the Lagrangian which restores the invariance of the action under the boundary-preserving subalgebra. Suppose that we had disregarded the "non-physical" components of $\mathscr{L}$ from the start and concerned ourselves solely with finding corrections to the "physical" L component. In principle δL could have been derived term by term, with each term chosen to cancel the inadequacies of the last. However, it is not difficult to see that the same result would have been obtained in the end, although in a less transparent form.

In general, it is a straightforward exercise to verify that (13) and (14) do yield invariant actions for examples of interest. The special case $n = 1$, $D = 2$, which corresponds to simple supersymmetric quantum mechanics (Cooper and Freedman, 1983), has been considered (Luckock, 1991). (In this case there are only two subalgebras of the supersymmetry algebra which preserve the boundary.) Similarly, the case $n = 1$, $D = 4$ has been considered by Bollé *et al.* (1990), who obtained results of the same kind by an exhaustive analysis of the possible forms of the Lagrangian. Expression (13) also reproduces known results in the cases $n = 2$, $D = 2$ and $n = 3$, $D = 2$, which are relevant to models of supersymmetric strings and membranes (Luckock, 1989; Luckock and Moss, 1989).

3. INVARIANT BOUNDARY CONDITIONS

Given a super-Lagrangian $\mathscr{L}(x^\mu, \theta^a)$ which is functionally dependent on some superfield

$$\Phi(x^\mu, \theta^a) = \phi(x) + \theta^a \psi_a(x) + \tfrac{1}{2}\theta^a\theta^b\lambda_{ab}(x) + \cdots \quad (15)$$

we can use the results of the previous section to write down an action $S[\Phi]$

which is invariant under the boundary-preserving subalgebra. In this context, it is natural to ask whether it is possible to impose boundary conditions on Φ which are invariant under this subalgebra.

We begin by looking for boundary conditions which transform in a well-defined manner under the action of the generators $Q_{\hat{a}}$. The problem is that the transformations of most components of Φ are determined by the derivatives of other components. The components which depend on the coordinates $\theta^{\bar{a}}$ are a particular nuisance, since their variations depend on derivatives normal to the boundary; for example, if ϵ is a constant anticommuting spinor parametrizing an infinitesimal boundary-preserving supersymmetry transformation, then

$$\delta_\epsilon \psi_{\bar{a}} = -\epsilon^{\hat{b}} \lambda_{\hat{b}\bar{a}} - \epsilon^{\hat{b}} (C\gamma^\mu)_{\hat{b}\bar{a}} \partial_\mu \phi \tag{16}$$

In order to know how a boundary condition on $\psi_{\bar{a}}$ should transform, one must therefore impose various boundary conditions on ϕ, including one on its derivative normal to the boundary. The problem becomes worse if we wish to impose boundary conditions on components which depend on more than one $\theta^{\bar{a}}$.

To avoid such difficulties, let us project out a "reduced superfield" $\hat{\Phi}(x^\mu, \theta^{\hat{a}})$ by eliminating all components of Φ which depend on the coordinates $\theta^{\bar{a}}$; thus

$$\hat{\Phi}(x^\mu, \theta^{\hat{a}}) = \phi(x) + \theta^{\hat{a}} \psi_{\hat{a}}(x) + \tfrac{1}{2} \theta^{\hat{a}} \theta^{\hat{b}} \lambda_{\hat{a}\hat{b}}(x) + \cdots \tag{17}$$

One then finds that the components of $\hat{\Phi}$ transform into each other under the action of the generators $Q_{\hat{a}}$. Any boundary conditions imposed on these components will therefore transform in a well-defined manner under the action of the boundary-preserving subalgebra.

In fact, the generators $\hat{Q}_{\hat{a}}$ act on the reduced superfield $\hat{\Phi}$ to give the same transformation laws for its components as do the generators $Q_{\hat{a}}$. It follows that the boundary-preserving subalgebra (4) can be represented by the action of the reduced supersymmetry algebra (8) on $\hat{\Phi}$.

A set of boundary conditions on the components of the reduced superfield $\hat{\Phi}$ is not usually invariant under the boundary-preserving subalgebra. However, the orbit of such a set, under the action of the corresponding group, obviously is. We can therefore recover invariance by merely requiring that the reduced superfield $\hat{\Phi}$ should obey *any* of the boundary conditions in such an orbit. In other words, we impose boundary conditions which need only be obeyed up to a possible reduced supersymmetry transformation. In fact, boundary conditions of exactly this sort have been used in the context of supersymmetric strings and membranes (Luckock, 1989; Luckock and Moss, 1989).

4. CONCLUSION

If an n-dimensional model has a supersymmetry which admits a boundary-preserving subalgebra with $\hat{n}$ bosonic generators, then we can use the methods described in Section 2 to obtain a type of reduced supersymmetry in $\hat{n}$ dimensions. In particular, we can add to the action a boundary correction which restores part of the supersymmetry broken by the boundary.

In quantum mechanical applications, the boundary often consists of two pieces which represent initial and final hypersurfaces. In this context, transition amplitudes are defined by summing over all field configurations obeying specified boundary conditions, which correspond to the initial and final states. Using the results of Sections 2 and 3, we can obtain an action functional and boundary conditions which are invariant under the boundary-preserving subalgebra. When these are employed, the resulting amplitudes will be invariant under the reduced supersymmetry generated by this subalgebra.

One problem that has not yet been addressed is that of identifying the boundary-preserving subalgebra. On a case-by-case basis, if one exists, then it can usually be found quite easily. In fact, when the boundary is flat, the subalgebra can often be enlarged to include the generators of boundary-preserving rotations. The reduced supersymmetry obtained from this subalgebra is then a bona fide supersymmetry in $\hat{n}$ dimensions.

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